Representation Theory of the Parity Moduli of the 24-Cell Flag Complex

Admissible rotational actions, the intrinsic 3+4 orbit decomposition, and the algebraic constraints on equivariant realizations

Charles Richard Walker (C. Rich)
Independent Researcher / mylivingai.com
ORCID: 0009-0007-6541-3905
Cosmological Pangaea Project: osf.io/vf5cw

Abstract

The flat discrete Z_2-connection constructed on the 576-flag complex of the regular 24-cell yields a moduli space of global sections isomorphic to $(Z_2)^3$ under the discrete gauge formulation of the preceding paper. This note classifies the admissible actions of the classical finite rotation groups on that moduli space. The only groups among the cyclic, dihedral, tetrahedral, octahedral and icosahedral families (and their binary covers) that admit nontrivial homomorphisms into $\mathrm{Aut}((Z_2)^3) = GL(3,2)$ are those without an A_5 factor. For the admissible groups A_4 and S_4 the natural linear action on $V = F_2^{\,3}$ is faithful, fixes the origin, and decomposes the seven nonzero vectors into two orbits of sizes 3 and 4. The corresponding stabilizers are V_4 and C_3 (for A_4) and D_4 and S_3 (for S_4). Any subsequent A_4- or S_4-equivariant geometric realization of the parity moduli is therefore forced to preserve this orbit decomposition.

1. Dependence and Scope

This paper assumes the results of the preceding two works in the series:

• the finite combinatorial construction of the 24-cell flag complex, the move-based signed Z_2 system, the three balanced connected components, the eight global solutions, and the two $W(F_4)$ orbits (Paper I);

• the interpretation of that system as a flat discrete Z_2-connection whose moduli space, under the discrete gauge formulation defined therein, is identified with $H^1(G_{move}; Z_2)$ ■ $(Z_2)^3$ (Paper II).

All representation-theoretic statements concern the action induced on this three-dimensional vector space over F_2. No continuum geometry or celestial embedding is assumed or claimed.

2. Admissible Rotational Symmetries

The automorphism group of the moduli space is $\mathrm{Aut}((Z_2)^3) = GL(3,2)$ ■ $PSL(2,7)$, of order 168. Because the target is finite, only finite groups can act nontrivially.

Among the classical finite subgroups of SO(3) and their binary covers in SU(2), the groups that admit a nontrivial homomorphism into GL(3,2) are precisely those that do not contain an A_5 (or $2A_5$) factor. Explicitly:

• cyclic and suitable dihedral groups embed when the relevant element orders divide 168;
• the tetrahedral group A_4 and the octahedral group S_4 embed;
• the icosahedral group A_5 and its binary cover do not embed (order considerations and simplicity).

Thus the only polyhedral symmetries available for an equivariant geometric realization are tetrahedral and octahedral.

3. The Admissible Linear Actions

Concrete matrix realizations of the embeddings A_4 ■ GL(3,2) and S_4 ■ GL(3,2) were obtained by generating the groups from explicit generators over F_2. Both actions are faithful.

Theorem (Orbit Structure of the Admissible Representation).

Let $V = F_2^3$ carry the faithful representation of the embedded A_4 (respectively S_4) obtained from the natural action on the moduli space $(Z_2)^3$. Then:

1. The origin $0 \in V$ is fixed.
2. The set of nonzero vectors decomposes into exactly two orbits

$$V \setminus \{0\} = O_3 \blacksquare O_4$$

with $|O_3| = 3$ and $|O_4| = 4$.

3. The stabilizers are

$$\mathrm{Stab}_{A_4}(O_3) \blacksquare V_4, \mathrm{Stab}_{A_4}(O_4) \blacksquare C_3$$

and

$$\mathrm{Stab}_{S_4}(O_3) \blacksquare D_4, \mathrm{Stab}_{S_4}(O_4) \blacksquare S_3.$$

Proof. The orders of the stabilizers follow at once from the orbit–stabilizer theorem once the orbit sizes are known. Direct computation of the stabilizers of representatives of each orbit yields groups of the indicated isomorphism types (Klein four-group, cyclic of order 3, dihedral of order 8, and symmetric group of degree 3, respectively). Faithfulness is verified by confirming that the kernels of the constructed homomorphisms are trivial.

Corollary.

Any A_4- or S_4-equivariant geometric realization of the parity representation necessarily preserves the orbit decomposition $O_3 \blacksquare O_4$. Consequently three parity directions possess strictly larger stabilizer subgroups than the remaining four.

4. Structural Interpretation

Before the imposition of admissible rotational symmetry the moduli space appears as three independent binary degrees of freedom. After the symmetry is imposed the seven nonzero linear functionals are fused into two irreducible symmetry classes distinguished by stabilizer type. The three directions belonging to O_3 are stabilized by larger subgroups and are therefore the more symmetric parity combinations; the four directions belonging to O_4 are less symmetric. This 3+4 splitting is an intrinsic feature of the admissible representation, independent of any particular choice of matrix generators within a conjugacy class.

5. Consequences for the Geometric Stage

The algebraic constraints that any subsequent equivariant embedding must satisfy are now completely known:

• the induced action on the parity moduli must realize one of the classified embeddings of A_4 or S_4;
• the embedding must preserve the orbit decomposition $O_3 \blacksquare O_4$;
• the stabilizers of the two classes of parity directions must be realized geometrically as the groups listed in the theorem.

These conditions are necessary. Whether they are also sufficient is the content of the geometric existence problem formulated in the preceding paper and left open here.

6. Computational Reproducibility

All matrix generators, orbit computations, and stabilizer calculations were performed independently over F_2 and will be archived with the series so that every step is verifiable.

References

Walker, C. R. Cosmological Pangaea: Constraint Closure and Symmetry Reduction on the 24-Cell Scaffold (Paper I). OSF, 2026.

Walker, C. R. Flat Z_2 Connections on the 24-Cell Flag Complex: The Celestial Embedding Problem (Paper II). OSF, 2026.
Standard references on GL(3,2), PSL(2,7), and the finite subgroups of SO(3) and SU(2).