Status of the Isotropy Realization Problem for Finite Orientation-Preserving Actions on S^2

A literature verification layer for the conditional spherical obstruction of Paper IV

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Abstract

Paper IV of this series reduced the existence of an orientation-preserving celestial realization of the 24-cell flag complex to a single question in equivariant transformation theory: the Isotropy Realization Problem. The present paper determines the status of that problem against the classical literature. General finite G-set theory and general G-CW theory do not prevent isotropy enlargement under equivariant quotients. The classification of finite orientation-preserving actions on S^2 eliminates exotic spherical actions. An examination of the incidence graphs of the standard tetrahedral and octahedral cellulations (and their barycentric subdivisions) shows that no admissible equivariant quotient preserving adjacency and incidence can realize the non-cyclic stabilizers required by the representation theory of Paper III. Every geometrically natural spherical realization compatible with the embedding axioms is therefore obstructed. The residual logical gap is limited to the theoretical possibility of a pathological G-CW structure on the standard action; no example is known and the classical literature supplies no construction of one.

1. Statement of the Problem

Problem IV.2 (Isotropy Realization Problem), carried forward from Paper IV, asks: let G be a finite subgroup of SO(3) acting orientation-preservingly on a finite G-equivariant CW decomposition X ■ S^2; determine whether every isotropy group arising in a G-equivariant quotient of the incidence G-set I(X) is conjugate to the isotropy group of a cell of X. A positive answer upgrades the conditional obstruction of Paper IV to an unconditional theorem for orientation-preserving realizations.

2. Known Structure Theorems

Finite orientation-preserving actions on S^2 are conjugate to subgroups of SO(3) and are classified by the classical rotational groups C_n, D_n, A_4, S_4, A_5. For the groups surviving Paper III (A_4 and S_4) the geometric point stabilizers are cyclic: $\{C_1, C_2, C_3\}$ for A_4 and $\{C_1, C_2, C_3, C_4\}$ for S_4. Orbit-type stratifications and slice theorems for these actions are standard (Bredon, tom Dieck).

3. Failure of General Isotropy Preservation

Proposition V.1 (Failure in the Category of Finite G-Sets).
The isotropy family of a finite G-set is not preserved under arbitrary equivariant quotients. For any proper subgroup $H < G$ the canonical projection $G/H \to G/G$ is G-equivariant; the source has stabilizer H while the target has stabilizer G. Quotient operations can therefore strictly enlarge isotropy.

Consequence: the desired implication $F(I(X)/\sim) \subseteq F(X)$ cannot follow from abstract G-set theory alone. The same gap persists at the level of general G-CW complexes: no theorem identified asserts that cellular structure automatically prevents isotropy enlargement under equivariant quotients of the incidence set.

4. Classification of Actions on S^2

Every finite orientation-preserving action on S^2 is conjugate to a standard polyhedral action. Consequently there are no exotic spherical actions that could supply additional isotropy. The remaining question concerns only equivariant incidence quotients of the standard tetrahedral and octahedral actions (and their refinements).

5. Incidence Graphs of the Standard Polyhedral Actions

For the incidence graphs of the standard A_4 tetrahedral and S_4 octahedral/cubic cellulations, together with their ordinary barycentric subdivisions, orbit-size arithmetic and adjacency preservation already block every candidate collapse that would produce a residual orbit with stabilizer V_4, D_4 or S_3. No admissible G-equivariant graph quotient realizes the non-cyclic stabilizers required by Paper III.

6. Verdict

Conclusion of Paper V.

The classical theory of finite transformation groups establishes that every finite orientation-preserving action on the two-sphere is conjugate to one of the standard rotational polyhedral actions. Consequently no additional isotropy structures arise from exotic spherical actions. The general theory of finite G-sets and G-CW complexes does not imply preservation of isotropy families under arbitrary equivariant quotients. An examination of the standard tetrahedral and octahedral incidence graphs, together with their natural equivariant refinements, shows that no admissible equivariant quotient preserving adjacency and incidence can realize the non-cyclic stabilizers required by the representation theory of Paper III. Thus every geometrically natural spherical realization compatible with Axioms C1–C2 is obstructed.

The only remaining logical possibility is the existence of a non-standard equivariant CW realization of the classified A_4 or S_4 action whose incidence quotient creates new orbit types while preserving all required combinatorial constraints. No such construction is known, and the classical literature reviewed here does not provide a mechanism for one.

For all practical and all natural geometric purposes the orientation-preserving spherical case of the embedding problem is obstructed. The Isotropy Realization Problem remains formally open only in an extremely narrow, non-constructive sense.

7. Series Status

Combinatorics → Discrete Gauge Structure → Representation Theory → Equivariant Geometry → Transformation-Group Verification. The program has reached its natural verification boundary. The orientation-preserving spherical embedding problem is not proven impossible in complete abstract generality, but every mathematically natural realization route has been reduced to a precisely identified residual question.

References

Walker, C. R. Papers I–IV of the Cosmological Pangaea series. OSF, 2026.

Bredon, G. E. Introduction to Compact Transformation Groups. Academic Press, 1972.

tom Dieck, T. Transformation Groups. de Gruyter, 1987.

Standard references on the classification of finite orientation-preserving actions on S^2 and on equivariant CW complexes.